Laplace T/F

A function that converts a real time signal to complex frequency domain.

- signal always starts at or after $t=0$
- signal is defined for every instant (continuous)

Mathematically

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad s = j\omega$$

$$s = \delta + j\omega$$

δ = decay in signal
ω = frequency of signal

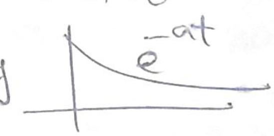
$$\boxed{f(t) = 0 \quad \forall\, t < 0}$$

▷ Laplace transform must have a countable - finite integral

$$\int_0^\infty e^{-at}\,|f(t)|\,dt < \infty$$

where a is real number and is constant

so $e^{-\text{constant} * t} \Rightarrow \frac{1}{e^{at}}$ is always decreasing



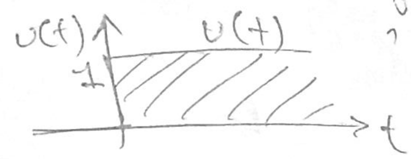
▷ e^{t^2} has no LT (it grows with time)

Inverting a LT function

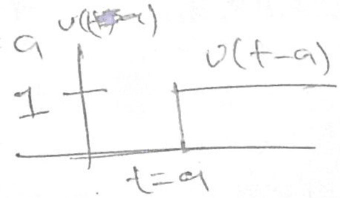
$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\delta_1 - j\infty}^{\delta_1 + j\infty} F(s)e^{st}\,ds.$$

Unit step Function

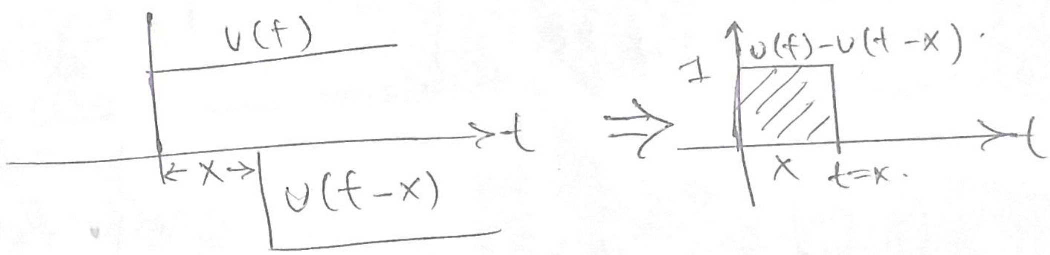
$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$u(t-a) = \begin{cases} 1 & t-a=0,\ t=a \\ 0 & \text{elsewhere} \end{cases}$$



Composing an impulse of width x.



$u(t) - u(t-x)$.

LT of step fxn

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mathcal{L}(u(t)) = \int_0^\infty u(t) e^{-st}\, dt$$

$$= \int_0^\infty 1\, e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty = -\frac{1}{s}\left(e^{-\infty} - e^{-0}\right) = -\frac{1}{s}\left(\frac{1}{\infty} - 1\right)$$

$$= -\frac{1}{s}(-1) = \frac{1}{s}$$

$$\boxed{\mathcal{L}(u(t)) = F(s) = \frac{1}{s}}$$

$$\mathcal{L}(u(t-a)) = \int_0^\infty u(t-a)\, e^{-st}\, dt$$

$$= \int_a^\infty 1\, e^{-st} = -\frac{1}{s}\left(e^{-\infty} - e^{-as}\right)$$

$$= +\frac{e^{-as}}{s} \qquad \boxed{s > 0} \text{ converging}$$

so

②

$F(s)$ for $u(t) - u(t-a)$

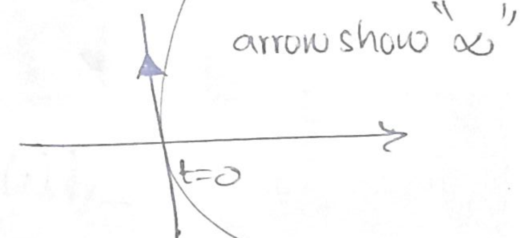
$$\mathcal{L}(u(t) - u(t-a)) = \mathcal{L}(u(t)) - \mathcal{L}(u(t-a))$$

$$= \frac{1}{s} - \frac{e^{-sa}}{s} = \frac{1 - e^{-as}}{s} \quad s > 0$$

An impulse function

- it is used for sampling
- its value is defined for only one instant
- denoted by $\delta(t)$.

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{elsewhere} \end{cases}$$



$$\mathcal{L}(\delta(t)) = \int_0^\infty \delta(t) e^{-st} \, dt$$

$$= \int_0^\infty [u(t) - u(t-x)] e^{-st} \, dt$$

$x \to 0$ so t and $t - x$ are same place

$$= \int_0^{0+\epsilon} 1 e^{-st} \, dt$$

$$\mathcal{L}(\delta(t)) = 1 \quad ; \text{ as } e^{-s0} = e^0 = 1$$

Note: $\delta(t)$ is a ~~[illegible]~~ rectangle $\ni$ area $= 1$

its width $= 0$ and its height $= \infty$

Mathematically : $1 = 0 * \infty = \frac{1}{\infty} * \infty$

---

if impulse is at some other instant t_0 rather than being at $t = 0$ then

Laplace TF

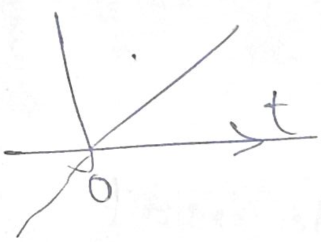
$$\mathcal{L}(\delta(t - t_0)) = \begin{cases} 1 & \text{at } t = t_0 \\ 0 & \text{else where} \end{cases}$$

ramp Function

it has a slope that depends upon "t"

$$f(t) = t$$

$$\mathcal{L}(f(t)) = \int_0^\infty t e^{-st}\, dt$$



$$= t\int_0^\infty e^{-st}\,dt - \int\int e^{-st}\, dt \frac{d}{dt} t \, dt$$

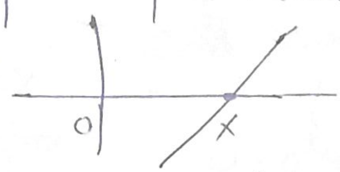
$$= t\frac{e^{-st}}{-s}\Big|_0^\infty - \int \frac{e^{-st}}{-s}(1)\, dt$$

$$= t\frac{e^{-st}}{-s}\Big|_0^\infty + \frac{e^{-st}}{-s^2}\Big|_0^\infty$$

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{s^2}}$$

if f(t) is delayed by a value 'X' then

new $f(t-x)$



$$\mathcal{L}(f(t-x)) = \frac{1}{(s-x)^2}$$

A cosine Function

$$f(t) = \cos(wt)$$

$$\mathcal{L} f(t) = \int_0^\infty \cos wt\, e^{-st}\, dt$$

using $\int_0^\infty 1 e^{-st} \to \frac{1}{s}$

$$= \int_0^\infty \frac{e^{jwt} + e^{-jwt}}{2} e^{-st}\, dt$$

$$= \int_0^\infty \frac{e^{-(s-wj)t} + e^{-(s+jw)t}}{2}$$

$$= \frac{1}{2}\left(\frac{1}{s-jw} + \frac{1}{s+jw}\right)$$

$$\mathcal{L}(\cos wt) = \frac{s}{s^2+w^2}$$

Laplace of Mixed (product) signals (3)

rules 1: e^{-at} produces a delay of $\frac{1}{s-a}$ in s^{th} term

2: $\frac{d}{dt}$ produces an s in multiplication

3: $\int dt$ produces $\frac{1}{s}$ in multiplication

4: The power of ramp function is converted to $\frac{1}{s^{(power+1)}}$

5: a delay in time, say $(t-t_0)$ will produce $\boxed{e^{-t_0 s}}$ in multiplication

6: $\int_0^t f(t-t_0)\,dt$ is same as $\int_0^t f(t)\,dt$

Example 13.5

$$f(t) = e^{-at}\cos wt$$

$$\mathcal{L}(\cos wt) = \frac{s}{s^2+w^2}$$

$$\boxed{\mathcal{L}(e^{-at}\cos wt) = \frac{s+a}{(s+a)^2+w^2}}$$

$$f(t) = te^{-at}$$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$$\boxed{\mathcal{L}(te^{-at}) = \frac{1}{(s+a)^2}}$$

$$\int^t y(\lambda)e^{-2(t-\lambda)}$$

$t-\lambda=0$

$\lambda=t$

$$\int_0^{\lambda} y(t)\,e^{-2(t-\lambda)}\,dt$$

13.6. solving differential equation with Laplace

$$\frac{dy(t)}{dt} + 2y(t) + \int_0^t y(\lambda)\underbrace{e^{-2(t-\lambda)}}_{\text{delay}}\,d\lambda = 10u(t) \qquad y(0)=0$$

$$sY(s) + 2Y(s) + \frac{Y(s)}{s+2} = \frac{10}{s}$$

$$Y(s) = \frac{10(s+2)}{s(s^2+4s+5)}$$